Operator analysis of $\sigma(e^+e^- \to X)$

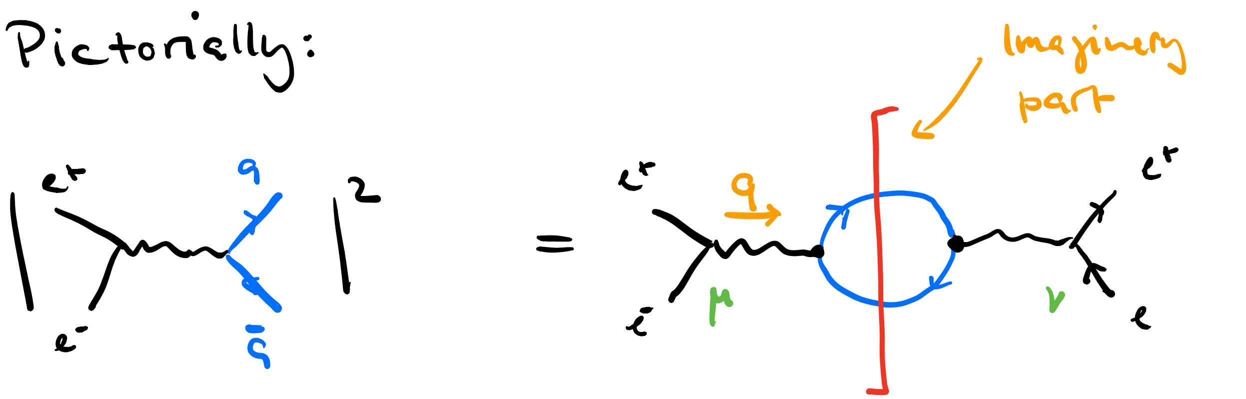
Our simple LO computation for the R-ratio lead to quite impressive agreement with experimental data, but there are aspects of the computation, which are unsatisfactory. We have shown that we get finite results computing in perturbation theory, but why is it OK to compute with unphysical quark & gluon states? There should be hadronisation corrections: how big are they?

For the total cross section, there are answers to these questions obtained after combining the optical theorem with the Operator Product Expansion (OPE).

The optical theorem states that one can obtain the total cross section from the imaginary part of the forward amplitude. Applied to our case, it reads (exercise)

$$\sigma(e^+e^- \to X) = \frac{1}{s} \,\mathrm{Im}\, \mathcal{M}(e^+e^- \to e^+e^-)$$

Pictorially:



One can rewrite this in slightly nicer form by noticing that the intermediate piece in the forward scattering is just the hadronic part of the two-point function of the photon, i.e.

Follows from Ward identity $q^\mu \Pi_{\mu\nu} = 0$.

$$\Pi^h_{\mu\nu}(q^2) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi^h(q^2).$$

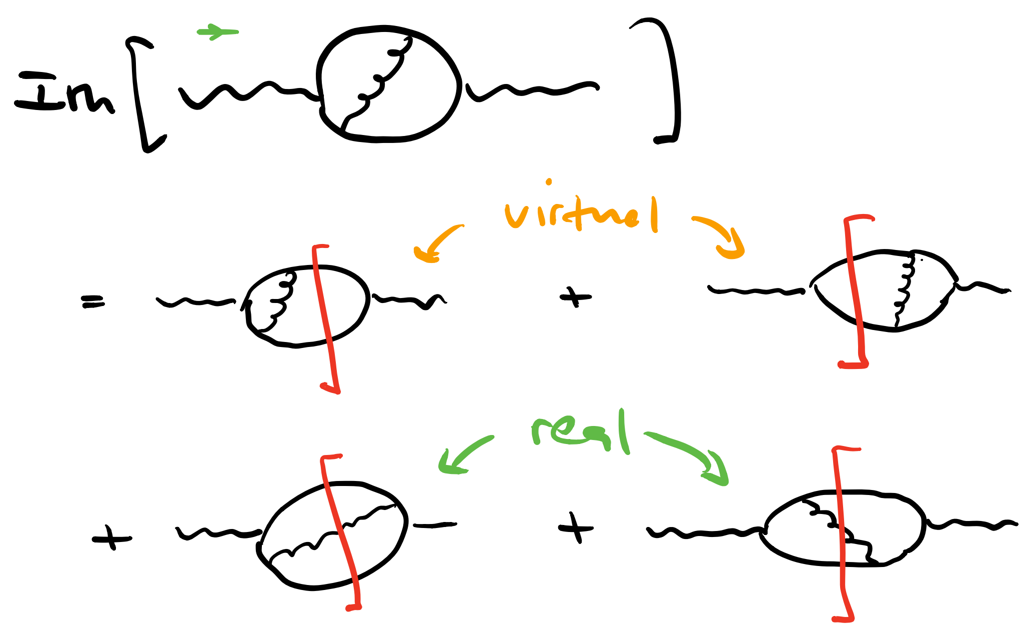
Plugging this into the representation for the forward amplitude yields

$$\boxed{\sigma(e^+e^- \to \text{hadrons}) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(s)} \quad (*)$$

This relation plays an important role in the computation of the muon $g-2$: one uses the measurement of σ to get information on the hadronic part of the self-energy.

Here we use the relation in the opposite way: we will analyze the two point function to learn more about σ.

Note that the representation (*) makes the cancellation of IR singularities quite transparent:



Note: Imaginary parts arise when particles go on shell. Cutkosky rule: (exercise)

$$\frac{i}{p^2-m^2} \longrightarrow \Theta(p^0)(2\pi)\delta(p^2-m^2)$$

turns loop integrals into phase space integrals!

Making use of a general theorem, which states that off-shell Green's functions are IR finite, we also immediately get the result that σ is IR finite to all orders in perturbation theory.

The electromagnetic current of the quarks reads

$$J_\mu = \sum_f e_f \bar{\psi}_f \gamma^\mu \psi_f$$

and the hadronic part of the self-energy is

$$i\Pi^{\mu\nu}(q) = (ie)^2 \int d^4x \, e^{-iqx} \langle \Omega | T\{ J^\mu(x) J^\nu(0) \} | \Omega \rangle$$

vacuum

So it is given by a product of two current operators. We are interested in the cross section at large $Q^2 = q^2$, which corresponds to small x-values in the Fourier integral.

Operator product expansion (OPE) (Wilson '69)

The idea behind the OPE is that at small distance $x \to 0$, a product of operators can be expanded into a series of local operators

$$\mathcal{O}_A(x)\,\mathcal{O}_B(0) = \sum_i C_i(x^2)\,O_i(0) \sim E^d$$

the terms on the rhs are then ordered by their operator dimension. By dimensional analysis, if $d = d_A + d_B$ then

$$C_i(x^2) = c_i \cdot (x^2)^{(d_i - d)/2}$$

dimensionless

⇝ Higher-dim. operators are suppressed by powers of x^2!

In QFT, the dimensionless coefficients c_i still exhibit logarthic dependence on x^2 because we need to renormalize the local operators $O_i(0)$:

$$\rightsquigarrow \quad C_i \equiv C_i\big(\underset{\text{coupling}}{g(\mu)}, \underset{\text{renormalization scale.}}{\mu^2} x^2\big)$$

Let us look at an example in ϕ^4 - theory:

dominant at small x !

$$\underbrace{\phi(x)\phi(0)}_{d=2} = \underline{\frac{C_0}{x^2} \mathbb{1}} + C_2 \, \phi^2(0)$$

$$+ x^2 C_{4a} \, \partial_\mu \phi \partial^\mu \phi$$

$$+ x^2 C_{4b} \, \phi^4 \qquad + \dots$$

This is an operator relation. To determine the coefficients C_i one takes appropriate matrix elements.

Let us now apply this technique to the product of electromagnetic currents. Given the transversality of $\Pi_{\mu\nu}$, it is good enough to consider $\Pi_{\mu\nu} g^{\mu\nu}$ and expand

$$J_\mu(x) J^\mu(0) = \sum_i C_i(x^2) O_i(0)$$

Since we are only interested in the vacuum matrix element at the end of the day, it is sufficient to consider scalar operators, e.g.

$$C^{\mu\nu} \langle \Omega | \partial_\mu \phi \partial_\nu \phi | \Omega \rangle$$

$$\sim C^{\mu\nu} g_{\mu\nu} \langle \Omega | \partial_\alpha \phi \partial^\alpha \phi | \Omega \rangle .$$

From constructing the most general Lagrangian up to $d=4$, we then already know what the operators are:

$$\underbrace{i J_\mu(x) J^\mu(0)}_{d=6} = \overset{\swarrow \frac{1}{x^6}}{C_{\mathbb{1}}(x^2)}\, \mathbb{1} + \overset{\swarrow \frac{1}{x^2}}{C_{q\bar{q}}}\, m_f \underbrace{\bar{\psi}_f \psi_f(0)}$$

$$+ C_{G^2} \left(G^{\mu\nu}(0) \right)^2 + \ldots$$

convention, inserted i ~~because~~

$\Pi = + e^2 i J J$

$\frac{1}{x^2}$

violates chiral symm; comes with m_f

To obtain $\Pi_{\mu\nu}$, we then take the Fourier transform of this relation

$$\int d^4x \, e^{-iqx} J^\mu(x) J_\mu(0)$$

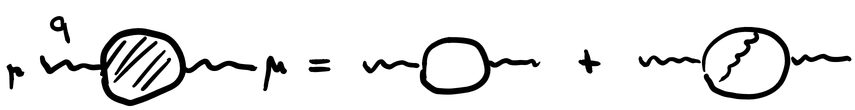
$$= \tilde{C}_{\mathbb{1}}(q^2) \cdot \mathbb{1} + \tilde{C}_{q\bar{q}}(q^2) \, m_f \bar{\psi}_f \psi_f$$

$$+ \tilde{C}_{G^2} (G_{\mu\nu})^2 + \ldots$$

Now we should determine the coefficients. Since the above equation is an operator relation, we can evaluate it with arbitrary external states, even unphysical ones, such as quarks & gluons. It is furthermore reasonable to expect that perturbation theory will be possible for the coefficients C_i since these describe short distance physics $x \to 0$ and $Q \to \infty$, respectively. Let us then evaluate the relation in perturbation theory for different external states:

1.) $|0\rangle$ (perturbative vacuum)

LHS:

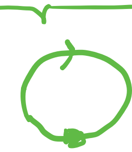


$+ \dots$

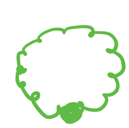
RHS:

$$\tilde{C}_1(q^2)\cdot \overbrace{\langle 0|\mathbb{1}|0\rangle}^{=1}$$

$$+\ \tilde{C}_{\bar{q}q}(q^2)\cdot \underbrace{\langle 0| m_f \bar{\psi}_f \psi_f |0\rangle}_{m_f \cdot}$$

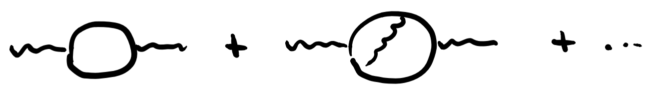


$$+\ \tilde{C}_G(q^2)\cdot$$



Then expand around $q^2 = \infty$, or equivalently $m_q = 0$, to find that

$$\tilde{C}_1(q^2) =$$

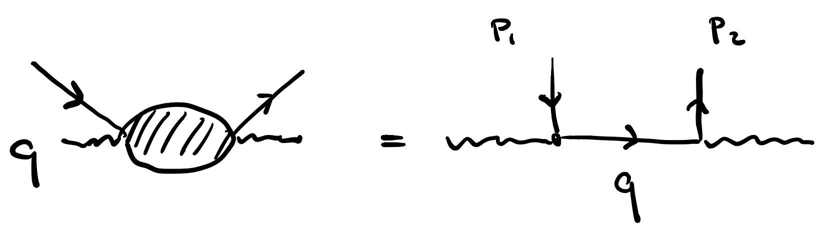


evaluated with $m_q = 0$.

Next we want to obtain $C_{\bar{q}q}$. To do so it is easiest to take a matrix element with external quarks:

2.)

LHS:



RHS:

$$C_1(q^2)\underbrace{\langle q(p_2)|q(p_1)\rangle}_{=0 \text{ for } p_1 \neq p_2}$$

$$+\, C_{\bar{q}q}(q^2)\, m_q\, \bar{u}(p_2)\, u(p_1) + C_G \cdot 0$$

$\Rightarrow$ this yields $\tilde{C}_{\bar{q}q}(q^2)$

To obtain $\tilde{C}_G$, we repeat the same with external gluons. Having determined the coefficients, we can now go back to relation (*) to obtain the cross section

$$\sigma(e^+e^- \to \text{hadrons}) = -\frac{4\pi\alpha}{s}\,\text{Im}\,\Pi_h(s)$$

we wrote the OPE for

$$g_{\mu\nu}\Pi_h^{\mu\nu} = g_{\mu\nu}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\Pi_h(q^2)$$

$$= 3q^2\,\Pi_h(q^2) = e^2\int d^4x\,\langle\Omega|T\{i\,j^\mu j_\mu\}|\Omega\rangle$$

so

$$\boxed{\sigma(e^+e^- \to \text{hadrons}) = -\frac{(4\pi)^2\alpha^2}{3s^2}\left(\text{Im}\left[\tilde{C}_1(q^2)\right]\overbrace{\langle\Omega|\Omega\rangle}^{=1}\right.}$$

$$+ \mathrm{Im}\left[\tilde{C}_{\bar{q}q}(q^2)\right] \underbrace{\langle \Omega | m_f \bar{\psi}_f \psi_f | \Omega \rangle}_{m_f B \ \dots\ \text{quark condensate} \ \sim \Lambda^3_{QCD} m_q}$$

$$+ \mathrm{Im}\left[\tilde{C}_G(q^2)\right] \underbrace{\langle \Omega | G^2_{\mu\nu} | \Omega \rangle}_{\text{Gluon condensate} \ \sim \Lambda^4_{QCD}}$$

By dimensional analysis, the contributions from $C_{\bar{q}q}$ & C_G are suppressed by $1/q^4$!

Our representation for σ, based on the OPE answers the questions posed at the beginning of the lecture:

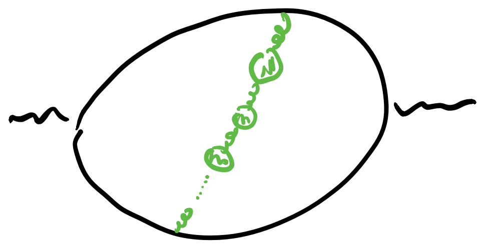
1.) The perturbative computation is correct in the limit $Q \to \infty$. OPE justifies the computation with unphysical states.

2.) Hadronisation corrections are suppressed by Q^4; the leading corrections are given by two non-perturbative matrix elements, the quark condensate and the gluon condensate.

The OPE representation also provides an example of factorization: it separates the low-energy non-perturbative physics from the perturbative high-energy physics.

It turns out that for $\sigma(e^+e^- \to x)$ the hadronisation effects are trivial in the limit $Q^2 \to \infty$. Not surprisingly, for pp or $e^- p$ collisions the situation is different. For these, one needs nontrivial matrix elements, already at leading power: the Parton Distribution Functions (PDFs).

Of course, our result is based on an expansion around $Q \to \infty$ and one can ask, wether this expansion is valid or even convergent. Asking for a convergent expansion in QFT is asking for too much. For example, one can show that the perturbative expansion of our correlator receives factorially large higher-order contributions from diagrams of the form (see e.g. Beneke hep-ph/9807443)



One can use a technique called Borel resummation to resum the series, but the result is ambiguous. It turns out that the ambiguity has the same size as the nonperturbative power corrections. So in some sense, perturbation theory is aware of the presence of nonperturbative effects.

Also the operator product expansion, i.e. the expansion in Q^2 and is missing terms of the form

$$\sim \frac{1}{(\rho Q)^n} e^{-\sqrt{Q^2}\rho}$$

in Euclidean space. Such terms will correspond to

$$\sim \frac{1}{(\rho Q)^n} e^{iQ\rho} \quad (!)$$

in Minkowski space. Resonance and instanton models exhibiting such behavior were studied e.g. by Shifman, QCD@work 2003 lectures.